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Strong Converse Inequality for the Bernstein–Durrmeyer Operator

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An equivalence relation between the rate of approximation of Bernstein–Durrmeyer polynomials and an appropriate K-functional is established. The results are stronger than those known for Bernstein polynomials. The advantages of Bernstein–Durrmeyer polynomials, i.e., self-adjointness, communativity, and simple expansion by orthogonal polynomials, are used extensively. © 1993 Academic Press, Inc.

1. INTRODUCTION

The Bernstein–Durrmeyer operator (see [10, 3]) is given by

$$M_n(f, x) = (n+1) \sum_{k=0}^{n} P_{n,k}(x) \int_0^1 P_{n,k}(y) f(y)\, dy, \tag{1.1}$$

where

$$P_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

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We prove a strong converse inequality of type A, in the terminology of [8], that is, we show

$$\|M_n f - f\|_p \sim \inf\left(\|f-g\|_p + \frac{1}{n} \|(\varphi^2 g')'\|_p \right) \tag{1.2}$$

for $1 \leqslant p \leqslant \infty$ with $\varphi(x)^2 = x(1-x)$. For $1 < p < \infty$, we prove an analogue of (1.2) for the multivariate Bernstein–Durrmeyer operator introduced by Derriennic [4]. In the cases $p=1$ or $p=\infty$ and the higher dimensional analogue of (1.1), we prove a somewhat weaker result (that is, a strong converse inequality of type B in the terminology of [8]). Several recent articles [1, 2, 6] proved (among other results) converse inequalities for these operators that are obviously weaker than those in the present paper.

2. Notations and Survey of the Proof

The Multivariate Bernstein–Durrmeyer operator was introduced by Derriennic [4] as

$$M_n(f, x) = \frac{(n+d)!}{n!} \sum_{(\beta/n) \in T} P_{n,\beta}(x) \int_T P_{n,\beta}(u) f(u)\, du, \tag{2.1}$$

where $x,\ u \in R^d$ $(x = (x_1, ..., x_d))$, $\beta = (k_1, ..., k_d)$ with k_i integers, and $T = \{u : 0 \leqslant u_i, \sum_{i=1}^d u_i \leqslant 1\}$. The polynomial $P_{n,\beta}(u)$ is given by

$$P_{n, k_1, ..., k_d}(u_1, ..., u_d) \equiv P_{n,\beta}(u) = \frac{n!}{\beta!\,(n-|\beta|)!} u^\beta (1-|u|)^{n-|\beta|}, \tag{2.2}$$

where $\beta! = k_1! \cdots k_d!$, $u^\beta = u_1^{k_1} \cdots u_d^{k_d}$ $(u_i^{k_i} = 1$ if $k_i = u_i = 0)$, $|u| = \sum_{i=1}^d u_i$ and $|\beta| = \sum_{i=1}^d k_i$.

Many properties were proven about the operators $M_n f$ which are quoted as we use them. We define, following Derriennic [6],

$$P(D) = \sum_{i=1}^d \frac{\partial}{\partial x_i} x_i(1-|x|) \frac{\partial}{\partial x_i} + \sum_{i<j} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) x_i x_j \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \tag{2.3}$$

and recall that for $f \in C^2(T)$, it was proved in [5] that

$$n\{M_n(f, x) - f(x)\} \to P(D) f(x). \tag{2.4}$$

The operator $P(D)$ given by (2.3) and introduced in [6] may take other forms, as can be seen in [4, 2].

The main result of our paper is the equivalence

$$\|M_n f - f\|_p \sim \inf\left(\|f-g\|_p + \frac{1}{n} \|P(D)\, g\|_p \right), \tag{2.5}$$

which is proved in Theorem 6.3 for all d when $1 < p < \infty$ and for $d = 1, 2,$ and 3 when $p = 1$ and $p = \infty$. For $p = 1$ and $p = \infty$ and $d > 3$, a weaker result than (2.5) is valid. The proof follows from a Bernstein-type inequality

$$\|P(D)\, M_n^2 f\|_p \leqslant dn\, \|f\|_p, \qquad 1 \leqslant p \leqslant \infty \tag{2.6}$$

(Theorem 3.2), and an improved Voronovskaja-type result

$$\left\| M_n f - f - \frac{\alpha_n(d)}{2} P(D)[M_n f + f] \right\|_p$$

$$\leqslant \left(\frac{1}{4} \alpha_n(d)^2 + \frac{1}{2} \frac{1}{(n+1)^2} \right) \|P(D)^2 f\|_p \tag{2.7}$$

(Theorem 4.1). It is the interplay between the exact constants in (2.6) and (2.7) that implies (2.5) for $d = 1$, 2, and 3, and the estimate (2.6) depends on d ($\alpha_n(d)$ is asymptotically independent of d). For $1 < p < \infty$ we use the L_2 estimate

$$\|P(D)\, M_n^r f\|_2 \leqslant \frac{n}{\sqrt{r}} \|f\|_2 \tag{2.8}$$

(Theorem 5.1) and the Riesz–Thorin interpolation theorem to obtain

$$\|P(D)\, M_n^r f\|_p < \varepsilon(r)\, n\, \|f\|_p, \qquad 1 < p < \infty, \tag{2.9}$$

with $\varepsilon(r) = o(1)$ as $r \to \infty$. The inequality (2.9) together with (2.7) is sufficient to prove (2.5) for $1 < p < \infty$ and all dimensions d. We conjecture that (2.9) and hence (2.5) is valid for $p = 1$ and $p = \infty$ in all dimensions (see Remark 6.4).

We note that for $d = 1$, when $p = 1$ or $p = \infty$ we cannot replace the K-functional on the right hand side of (2.5) with $\omega_\varphi^2(f, t)_p$ (where $\varphi^2 = x(1-x)$). This follows since the K-functional on the right hand side of (1.2) and $\omega_\varphi^2(f, t)_p$ are not equivalent for $p = 1$ and $p = \infty$ while (1.2) holds. For $1 < p < \infty$ the above expressions are equivalent. Hence, even for higher dimensions an equivalence result with an expression generalizing ω_φ^2 will falter for $p = 1$ and $p = \infty$. We trust that an equivalence of sorts will be proved for $1 < p < \infty$, but that is beyond the scope of this paper and our knowledge. As an equivalence between the K-functional above and

$\omega_S^2(f, t)_p$ (see [9, Chap. 12]) is not true for all p and as the rate of convergence is equivalent to the above K-functional, it is that K-functional that is the appropriate measure for this paper.

3. Estimate of $\|P(D)\, M_n f\|_p$

It follows from Derriennic's research [6], detailed only for $d=1$ and $d=2$, that

$$\|P(D)^r\, M_n f\|_p \leqslant Cn^r\, \|f\|_p. \tag{3.1}$$

We need for $r=1$ the following better estimate on the constant C.

THEOREM 3.1. *For $f \in L_p(T)$, where T is the d-dimensional simplex given in Section 2, and for $P(D)$ given by (2.3), we have*

$$\|P(D)\, M_n f\|_p \leqslant 2\, dn\, \|f\|_p. \tag{3.2}$$

Proof. First we show that it is sufficient to prove (3.2) for $p=\infty$ (or $p=1$). Assume (3.2) for $p=\infty$. We take $g \in C^2(T)$ and $f \in L_1(T)$ and then use [2, Lemma 2.5]

$$P(D)\, M_n g = M_n P(D)\, g, \qquad g \in C^2(T). \tag{3.3}$$

We recall from [4] the self-adjointness of M_n and $P(D)$ with respect to the scalar product $\langle f, g\rangle = \int_T f(u)\, g(u)\, du$ to obtain

$$\begin{aligned} |\langle P(D)\, M_n f, g\rangle| = |\langle f, P(D)\, M_n g\rangle| &\leqslant \|f\|_{L_1(T)}\, \|P(D)\, M_n g\|_{L_\infty(T)} \\ &\leqslant 2\, dn\, \|f\|_{L_1(T)}\, \|g\|_{L_\infty(T)}. \end{aligned} \tag{3.4}$$

As (3.4) is valid for all $g \in C^2(T)$, we have (3.2) for $p=1$. The inequality (3.2) for $p=\infty$ and $p=1$ implies now (3.2) for $1<p<\infty$ via the Riesz–Thorin interpolation theorem.

We observe that

$$x_i(1-|x|)\frac{\partial}{\partial x_i} P_{n,\beta}(x) = (k_i(1-|x|)-(n-|\beta|)\, x_i)\, P_{n,\beta}(x), \tag{3.5}$$

and hence

$$\begin{aligned} &\frac{\partial}{\partial x_i} x_i(1-|x|)\frac{\partial}{\partial x_i} P_{n,\beta}(x) \\ &\quad = \frac{(k_i(1-|x|)-(n-|\beta|)\, x_i)^2}{x_i(1-|x|)} P_{n,\beta}(x) - (n-|\beta|+k_i)\, P_{n,\beta}(x). \end{aligned} \tag{3.6}$$

Similarly,

$$\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right) x_i x_j \left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right) P_{n,\beta}(x)$$
$$= \frac{(k_i x_i - k_j x_i)^2}{x_i x_j} P_{n,\beta}(x) - (k_i + k_j) P_{n,\beta}(x). \tag{3.7}$$

Recalling $M_n(1, x) = 1$, we have

$$\begin{aligned} 0 &= P(D)\, M_n(1, x) \\ &= \sum_{(\beta/n)\in T} \left(\left\{ \sum_{i=1}^{d} \frac{(k_i(1-|x|)-(n-|\beta|)\, x_i)^2}{x_i(1-|x|)} \right.\right. \\ &\quad \left.\left. + \sum_{i<j} \frac{(k_i x_j - k_j x_i)^2}{x_i x_j} \right\} - nd \right) P_{n,\beta}(x) \\ &\equiv \sum_{(\beta/n)\in T} (I_{n,\beta}(x) - nd)\, P_{n,\beta}(x), \end{aligned} \tag{3.8}$$

which implies

$$\sum_{(\beta/n)\in T} I_{n,\beta}(x)\, P_{n,\beta}(x) = nd \sum_{(\beta/n)\in T} P_{n,\beta}(x) = nd.$$

We now estimate

$$b_{n,\beta} \equiv \left| \frac{(n+d)!}{n!} \int_T f(x)\, P_{n,\beta}(x)\, dx \right| \leqslant \|f\|_{L_\infty(T)} \tag{3.9}$$

and use $I_{n,\beta}(x) \geqslant 0$ to obtain

$$\begin{aligned} |P(D)\, M_n(f, x)| &\leqslant \sum_{(\beta/n)\in T} (I_{n,\beta}(x) + nd)\, P_{n,\beta}(x)\, \|f\|_{L_\infty(T)} \\ &\leqslant 2nd\, \|f\|_{L_\infty(T)}. \quad \blacksquare \end{aligned}$$

We are also able to prove the following useful estimate.

THEOREM 3.2. *Under the assumptions of Theorem* 3.1, *we have*

$$\|P(D)\, M_n^2 f\|_p \leqslant dn\, \|f\|_p. \tag{3.10}$$

Proof. Following the proof of Theorem 3.1, we only have to consider $p = \infty$. We can write

$$
\begin{aligned}
|P(D)\, M_n^2(f, x)| &= |M_n P(D)\, M_n(f, x)| \\
&\leqslant \left(\frac{(n+d)!}{n!}\right)^2 \sum_{(\gamma/n)\in T} P_{n,\gamma}(x) \\
&\quad \times \sum_{(\beta/n)\in T} \left|\int_T P_{n,\gamma}(u)\, P(D)\, P_{n,\beta}(u)\, du\right| \\
&\quad \times \int_T P_{n,\beta}(v)\, |f(v)|\, dv \\
&\leqslant \frac{(n+d)!}{n!} \|f\|_{L_\infty(T)} \sum_{(\gamma/n)\in T} P_{n,\gamma}(x) \\
&\quad \times \sum_{(\beta/n)\in T} \left|\int_T P_{n,\gamma}(u)\, P(D)\, P_{n,\beta}(u)\, du\right|.
\end{aligned}
$$

We show

$$
\frac{(n+d)!}{n!} \sum_{(\beta/n)\in T} \left|\int_T P_{n,\gamma}(u)\, P(D)\, P_{n,\beta}(u)\, du\right| \leqslant nd, \tag{3.11}
$$

which implies (3.10) for $p=\infty$ and hence for $1 \leqslant p \leqslant \infty$. To prove (3.11), we write

$$
\begin{aligned}
J_{n,\gamma} &\equiv \frac{(n+d)!}{n!} \sum_{(\beta/n)\in T} \left|\int_T P_{n,\gamma}(u)\, P(D)\, P_{n,\beta}(u)\, du\right| \\
&= \frac{(n+d)!}{n!} \sum_{(\beta/n)\in T} \left|\sum_{i\leqslant j} \int_T (L_{i,j}(D)\, P_{n,\gamma}(u))(L_{i,j}(D)\, P_{n,\beta}(u))\, du\right|,
\end{aligned}
$$

where

$$
L_{i,i}(D) = \sqrt{u_i(1-|u|)}\, \frac{\partial}{\partial u_i}
$$

and

$$
L_{i,j}(D) = \sqrt{u_i u_j} \left(\frac{\partial}{\partial u_i} - \frac{\partial}{\partial u_j}\right) \qquad \text{for} \quad i \neq j. \tag{3.12}
$$

The straightforward computation of $L_{i,j}(D)\, P_{n,\eta}(u)$ (where $\eta=\beta$ or $\eta=\gamma$) leads now to

$$
\begin{aligned}
J_{n,\gamma} &\leqslant \frac{(n+d)!}{n!} \\
&\quad \times \sum_{(\beta/n)\in T} \int_T \left\{ \sum_{i=1}^{d} \frac{|k_i(1-|u|)-(n-|\beta|)\, u_i|\; |l_i(1-|u|)-(n-|\gamma|)\, u_i)|}{u_i(1-|u|)} \right. \\
&\quad \left. + \sum_{i<j} \frac{|k_i u_j - k_j u_i|\; |l_i u_j - l_j u_i|}{u_i u_j} \right\} P_{n,\gamma}(u)\, P_{n,\beta}(u)\, du.
\end{aligned}
$$

Recalling $I_{n,\eta}(u)$ (with $\eta=\beta$ and $\eta=\gamma$) given in (3.8), we use the Cauchy–Schwartz inequality to obtain

$$\begin{aligned}J_{n,\gamma} &\leqslant \frac{(n+d)!}{n!}\sum_{(\beta/n)\in T}\int_T I_{n,\beta}(u)^{1/2}\, I_{n,\gamma}(u)^{1/2}\, P_{n,\gamma}(u)\, P_{n,\beta}(u)\, du\\ &\leqslant \left\{\frac{(n+d)!}{n!}\sum_{(\beta/n)\in T}\int_T I_{n,\beta}(u)\, P_{n,\gamma}(u)\, P_{n,\beta}(u)\, du\right\}^{1/2}\\ &\quad\times\left\{\frac{(n+d)!}{n!}\sum_{(\beta/n)\in T}\int_T I_{n,\gamma}(u)\, P_{n,\gamma}(u)\, P_{n,\beta}(u)\, du\right\}^{1/2}\\ &\equiv J^*_{n,\gamma}\times J^{**}_{n,\gamma}.\end{aligned}$$

The estimate $J^*_{n,\gamma}\leqslant (nd)^{1/2}$ follows from

$$\sum_{(\beta/n)\in T} I_{n,\beta}(u)\, P_{n,\beta}(u)=nd,$$

which follows from (3.8). To estimate $J^{**}_{n,\gamma}$, we write, using (3.5),

$$\begin{aligned}\int_T \frac{(l_i(1-|u|)-(n-|\gamma|)\,u_i)^2}{u_i(1-|u|)}\, P_{n,\gamma}(u)\, du &= \int_T (l_i(1-|u|)-(n-|\gamma|)\,u_i)\\ &\quad\times\frac{\partial}{\partial u_i}P_{n,\gamma}(u)\, du\\ &=\frac{n!}{(n+d)!}(n-|\gamma|+l_i)\end{aligned}$$

and

$$\int_T \frac{(l_i u_j - l_j u_i)^2}{u_i u_j}\, P_{n,\gamma}(u)\, du = \frac{n!}{(n+d)!}(l_i+l_j),$$

which implies $J^{**}_{n,\gamma}\leqslant (nd)^{1/2}$. ▮

4. Voronovskaja-Type Estimates

Derriennic [5] proved the Voronovskaja-type estimate (2.4). For the converse inequality of the present paper, we need the following stronger result.

THEOREM 4.1. *Suppose $f\in C^4(T)$, $M_n f$ is given by (2.1) and $P(D)$ is given by (2.3). Then we have for $n>1$*

$$\left\| M_n f - f - \frac{\alpha_n(d)}{2} P(D)[M_n f + f] \right\|_p \leqslant \left(\frac{1}{4} \alpha_n(d)^2 + \frac{1}{2} \frac{1}{(n+1)^3} \right) \|P(D)^2 f\|_p \tag{4.1}$$

where

$$\alpha_n(d) \equiv \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} = \frac{1}{d} \left[\frac{1}{n+1} + \cdots + \frac{1}{n+d} \right].$$

Proof. Using Corollary 2.4 of [2],

$$M_n f - f = \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} P(D) M_k f,$$

we write

$$\begin{aligned} I(n) &\equiv \left\| M_n f - f - \frac{\alpha_n(d)}{2} P(D)(f + M_n f) \right\| \\ &= \frac{1}{2} \left\| \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} P(D)(M_k f - f) \right. \\ &\quad \left. + \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} P(D)(M_k f - M_n f) \right\| \\ &= \frac{1}{2} \left\| \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} \sum_{j=k+1}^{\infty} \frac{1}{j(j+d)} P(D)^2 M_j f \right. \\ &\quad \left. - \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} \sum_{j=n+1}^{k} \frac{1}{j(j+d)} P(D)^2 M_j f \right\| \\ &= \frac{1}{2} \left\| \sum_{j=n+2}^{\infty} \frac{P(D)^2 M_j f}{j(j+d)} \sum_{k=n+1}^{j-1} \frac{1}{k(k+d)} \right. \\ &\quad \left. - \sum_{j=n+1}^{\infty} \frac{P(D)^2 M_j f}{j(j+d)} \sum_{k=j}^{\infty} \frac{1}{k(k+d)} \right\| \\ &\leqslant \frac{1}{2} \sup_j \|P(D)^2 M_j f\| \sum_{j=n+1}^{\infty} \frac{1}{j(j+d)} \left| \sum_{k=n+1}^{j-1} \frac{1}{k(k+d)} - \sum_{k=j}^{\infty} \frac{1}{k(k+d)} \right| \end{aligned}$$

(with the understanding $\sum_{k=n+1}^{n} \cdots = 0$). Using Lemma 2.5 of [2], we have for $f \in C^4(T)$

$$P(D)^2 M_j f = M_j P(D)^2 f,$$

and hence,

$$\|P(D)^2 M_j f\| \leqslant \|P(D)^2 f\|.$$

We now have

$$I(n) \leqslant \frac{1}{2} \|P(D)^2 f\| \sum_{j=n+1}^{\infty} \frac{1}{j(j+d)} |\alpha_n(d) - 2\alpha_{j-1}(d)|$$
$$\equiv \frac{1}{2} \|P(D)^2 f\| \, J(n).$$

To estimate $J(n)$, we define j_0 by

$$j_0 = \max\{j: 2\alpha_{j-1}(d) - \alpha_n(d) > 0\},$$

and as $\alpha_j(d)$ is a decreasing sequence in j, we have

$$J(n) = \sum_{j=n+1}^{j_0} \frac{1}{j(j+d)} (2\alpha_{j-1}(d) - \alpha_n(d)) + \sum_{j=j_0+1}^{\infty} \frac{1}{j(j+d)} (\alpha_n(d) - 2\alpha_{j-1}(d))$$
$$\equiv J_1(n) + J_2(n).$$

To estimate $J_1(n)$, we write

$$J_1(n) = \sum_{j=n+1}^{j_0} (\alpha_{j-1}(d) - \alpha_j(d))(\alpha_{j-1}(d) + \alpha_j(d))$$
$$+ \sum_{j=n+1}^{j_0} \frac{1}{j^2(j+d)^2} - \alpha_n(d)(\alpha_n(d) - \alpha_{j_0}(d))$$
$$\leqslant \alpha_n(d)^2 - \alpha_{j_0}(d)^2 - \frac{1}{2}\alpha_n(d)^2 + \frac{2/3}{(n+1)^3}$$

as the definition of j_0 implies $\alpha_n(d) - \alpha_{j_0}(d) \geqslant \frac{1}{2}\alpha_n(d)$ and

$$\sum_{j=n+1}^{j_0} \frac{1}{j^2(j+d)^2} \leqslant \sum_{j=n+1}^{\infty} \frac{1}{j^2(j+d)^2} \leqslant \frac{2/3}{(n+1)^3} \qquad \text{for} \quad n \geqslant 1.$$

To estimate $J_2(n)$, we write

$$J_2(n) \leqslant \alpha_n(d)\, \alpha_{j_0}(d) - \sum_{j=j_0+1}^{\infty} (\alpha_{j-1}(d) - \alpha_j(d))(\alpha_{j-1}(d) + \alpha_j(d))$$
$$= \alpha_n(d)\, \alpha_{j_0}(d) - \alpha_{j_0}(d)^2.$$

Combining the estimates for $J_1(n)$ and $J_2(n)$, and as $j_0 \geqslant 2n+1$, we have

$$\begin{aligned} J(n) &\leqslant \frac{1}{2}\alpha_n(d)^2 + \frac{2/3}{(n+1)^3} + \alpha_{j_0}(d)(\alpha_n(d) - 2\alpha_{j_0}(d)) \\ &\leqslant \frac{1}{2}\alpha_n(d)^2 + \frac{2/3}{(n+1)^3} + 2\alpha_{j_0}(d)(\alpha_{j_0-1}(d) - \alpha_{j_0}(d)) \\ &\leqslant \frac{1}{2}\alpha_n(d)^2 + \frac{2/3}{(n+1)^3} + 2\frac{1}{(2n+2)^2}\frac{1}{2n+1} \leqslant \frac{1}{2}\alpha_n(d)^2 + \frac{1}{(n+1)^3}, \end{aligned}$$

which combined with the estimated of $I(n)$ concludes the proof. ∎

Remark 4.2. For most purposes, the slightly easier to prove estimate

$$\|M_n f - f - \alpha_n(d)\, P(D)\, f\|_p \leqslant \frac{1}{2n^2}\, \|P(D)^2 f\|_p \tag{4.2}$$

is sufficient. In some cases, however, (4.1) yields results which are qualitatively better.

In one result (Theorem 7.2), we need the following extension of (4.2).

THEOREM 4.3. *Suppose $f \in C^{2r+2}(T)$ and M_n, $P(D)$, T, and d are as given in Section 2. Then*

$$\|(M_n - I)^r f - \alpha_n(d)^r\, P(D)^r f\|_p \leqslant \frac{r/2}{n^{r+1}}\, \|P(D)^{r+1} f\|_p. \tag{4.3}$$

Proof. We first observe

$$\begin{aligned} \|M_n f - f - \alpha_n(d)\, P(D)\, f\|_p &= \left\| \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)}\, P(D)(M_k f - f) \right\|_p \\ &\leqslant \left\| \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} \sum_{j=k+1}^{\infty} \frac{1}{j(j+d)}\, P(D)^2\, M_j f \right\|_p \\ &\leqslant \sum_{k=n+1}^{\infty} \frac{1}{k(k+1)} \sum_{j=k+1}^{\infty} \frac{1}{j(j+1)}\, \|P(D)^2 f\|_p \\ &\leqslant \sum_{k=n+1}^{\infty} \frac{1}{k(k+1)^2}\, \|P(D)^2 f\|_p \\ &\leqslant \frac{1}{2}\frac{1}{n^2}\, \|P(D)^2 f\|_p. \end{aligned}$$

We prove (4.3) by induction. We assume (4.3) for $r = l$ and write

$$\|(M_n - I)^{l+1} f - \alpha_n(d)^l P(D)^l (M_n - I) f\| \leqslant \frac{l/2}{n^{l+1}} \|P(D)^{l+1} (M_n - I) f\|.$$

Since we have

$$\|P(D)^{l+1} (M_n - I) f\|_p = \|(M_n - I) P(D)^{l+1} f\|_p \leqslant \frac{1}{n} \|P(D)^{l+2} f\|_p$$

and since the induction hypothesis for $l = 1$ implies

$$\|\alpha_n(d)^l (M_n - I) P(D)^l f - \alpha_n(d)^{l+1} P(D)^{l+1} f\|_p \leqslant \frac{\alpha_n(d)^l}{2n^2} \|P(D)^{l+2} f\|_p,$$

the result follows. ∎

5. Estimate of $\|P(D) M_n^r f\|_2$ and Its Consequence

In this section, we will give an estimate of $\|P(D) M_n f\|_{L_2(T)}$ and of $\|P(D) M_n^r f\|_{L_2(T)}$ which will prove useful also for other $L_p(T)$.

Theorem 5.1. *Suppose $f \in L_2(T)$, $M_n f$, T and $P(D)$ are as defined in Section 2. Then we have $r = 1, 2, ...,$*

$$\|P(D) M_n^r f\|_{L_2(T)} \leqslant \frac{n}{\sqrt{r}} \|f\|_{L_2(T)}. \tag{5.1}$$

For the proof we need the following computational lemma.

Lemma 5.2. *For $\lambda_{n,k}$ given by*

$$\lambda_{n,k} = \frac{(n+d)!\, n!}{(n+d+k)!\, (n-k)!}, \qquad 0 \leqslant k \leqslant n, \tag{5.2}$$

we have

$$k(k+d)\, \lambda_{n,k}^r \leqslant n/\sqrt{r}, \qquad 0 \leqslant k \leqslant n. \tag{5.3}$$

Proof. Since $0 \leqslant \lambda_{n,k} \leqslant 1$, (5.3) follows immediately when $k(k+d) \leqslant n/\sqrt{r}$. To prove (5.3) for k satisfying $k(k+d) > n/\sqrt{r}$, we estimate $\lambda_{n,k}^j$ using

$$\lambda_{n,k}^j = \left(\frac{(n+d)!\, n!}{(n+d+k)!\,(n-k)!} \right)^j = \left(\prod_{i=1}^{k} \frac{n-k+i}{n+d+i} \right)^j = \prod_{i=1}^{k} \left(1 - \frac{d+k}{n+d+i}\right)^j$$

$$\leqslant \left(1 - \frac{d+k}{n+d+k}\right)^{kj} = \frac{1}{\left(1 + \frac{d+k}{n}\right)^{kj}} \leqslant \frac{1}{1 + k(k+d)\, jn^{-1}}.$$

For $j = 1$, we have

$$\lambda_{n,k} \leqslant \frac{n}{n + k(k+d)} \leqslant \frac{n}{k(k+d)}.$$

For $k(k+d) \geqslant n/\sqrt{r}$ and $j = r-1$, we have

$$\lambda_{n,k}^{r-1} \leqslant \frac{1}{1 + k(k+d)(r-1)\, n^{-1}} \leqslant \frac{1}{1 + (r-1)/\sqrt{r}} \leqslant \frac{1}{\sqrt{r}},$$

and hence,

$$k(k+d)\, \lambda_{n,k}^r \leqslant \frac{k(k+d)}{\sqrt{r}} \frac{n}{k(k+d)} \leqslant \frac{n}{\sqrt{r}}. \quad \blacksquare$$

Proof of Theorem 5.1. The eigenspaces of the self adjoint operators $P(D)\, f$ and $M_n f$ are the same (see **B**, (2.5) of [2], and Lemma 2.2 of [2]; see also [4]) and f can be expanded by

$$f = \sum_{k=0}^{\infty} P_k f,$$

where

$$M_n P_k f = \lambda_{n,k} P_k f \qquad \text{and} \qquad P(D)\, P_k f = -k(k+d)\, P_k f, \tag{5.4}$$

with $\lambda_{n,k}$ given by (5.2) for $k \leqslant n$ and $\lambda_{n,k} = 0$, $k > n$. We now have, using Bessel inequality and Parseval formula,

$$\|P(D)\, M_n^r f\|_{L_2(T)} = \left\| \sum_{k=1}^{n} k(k+d)\, \lambda_{n,k}^r\, P_k f \right\|_{L_2(T)}$$

$$= \left(\sum_{k=1}^{n} (k(k+d)\, \lambda_{n,k}^r)^2\, \|P_k f\|_{L_2(T)}^2 \right)^{1/2}$$

$$\leqslant \max_k \, (k(k+d)\, \lambda_{n,k}^r) \left(\sum_{k=1}^{n} \|P_k f\|_{L_2(T)}^2 \right)^{1/2}$$

$$\leqslant \frac{n}{\sqrt{r}} \|f\|_{L_2(T)}. \quad \blacksquare$$

The following estimate for $\|P(D)\, M_n f\|_p$ can now be derived.

COROLLARY 5.3. *For* $1 < p < \infty$ *and* $f \in L_p(T)$ *and any* $A > 0$, *there exists* r, $r = r(A, p, d)$, *such that*

$$\|P(D)\, M_n^r f\|_{L_p(T)} \leqslant An\, \|f\|_{L_p(T)}. \tag{5.5}$$

Proof. We recall that Theorem 3.1 implies

$$\|P(D)\, M_n^r f\|_{L_p(T)} \leqslant 2\, dn\, \|f\|_{L_p(T)}. \tag{5.6}$$

We now use the Riesz–Thorin interpolation theorem with (5.6) for $p = \infty$ (or $p = 1$) and (5.1) to obtain (5.5) for $2 \leqslant p < \infty$ (or $1 < p \leqslant 2$). ∎

6. Strong Converse Inequalities

In this section, we prove converse inequalities for the Bernstein–Durrmeyer operator. We duplicate some arguments from [8] for the sake of completeness. We define the K-functional

$$K_r(f, t^r)_p = \inf_{g \in C^{2r}(T)} (\|f - g\|_p + t^r\, \|P(D)^r\, g\|_p). \tag{6.1}$$

We note that in this section we are dealing with $r = 1$. We recall that

$$A_n \sim B_n \qquad \text{iff} \quad C^{-1} A_n \leqslant B_n \leqslant C A_n. \tag{6.2}$$

The converse result is given in the following theorem.

THEOREM 6.1. *Suppose* $P(D)$, $M_n f$ *and* T *are those given in Section* 2 *and* $K_1(f, t)_p \equiv K(f, t)_p$ *is given by* (6.1). *Then we have*

$$\|M_n f - f\|_p + \|M_{dn} f - f\|_p \sim K(f, 1/n)_p, \qquad 1 \leqslant p \leqslant \infty, \tag{6.3}$$

and

$$\|M_n f - f\|_p \sim K(f, 1/n)_p, \qquad 1 < p < \infty. \tag{6.4}$$

Remark 6.2. In the terminology of [8] the results (6.3) and (6.4) are strong converse inequalities of type B and A, respectively. Actually, for $d = 1$, (6.3) yields

$$\|M_n f - f\|_p \sim K(f, 1/n)_p \qquad \text{for} \quad 1 \leqslant p \leqslant \infty,$$

and this type of equivalence is shown for $d = 2$ and $d = 3$ as well (see

Theorem 6.3). For $d>1$, (6.3) has an advantage over (6.4) only for $p=1$ and $p=\infty$.

Proof. It was shown in (3.2) of [2] that

$$\|f-M_n f\|_p \leqslant 2K(f, n^{-1})_p,$$

and hence, we need only estimate $K(f, 1/n)_p$ by $\|M_n f-f\|_p+\|M_{nd} f-f\|_p$ or by $\|M_n f-f\|$ to prove (6.3) and (6.4), respectively. (Of course the conditions are not the same.) We do so by constructing $g \in C^2(T)$ such that both $\|f-g\|$ and $(1/n)\|P(D)\, g\|$ will satisfy the appropriate estimate. As the K-functional is given as an infimum on all $g \in C^2(T)$, we will have our result. To prove (6.3), we choose

$$g=\tfrac{1}{2}(M_{nd} M_n^2 f+M_n^2 f).$$

Using the commutativity relation $M_n M_m=M_m M_n$, we have

$$\begin{aligned}\|f-\tfrac{1}{2} M_{nd} M_n^2 f-\tfrac{1}{2} M_n^2 f\|_p &\leqslant \tfrac{1}{2}\|M_{nd} M_n^2 f-f\|_p+\tfrac{1}{2}\|M_n^2 f-f\|_p \\ &\leqslant \tfrac{1}{2}\|M_{nd} f-f\|_p+2\|M_n f-f\|_p .\end{aligned}$$

To estimate $P(D)\, g$, we use (4.1) but with nd rather than n, that is, we write

$$\begin{aligned}&\left\|M_{nd} \psi-\psi-\frac{\alpha_{dn}(d)}{2} P(D)(M_{nd} \psi+\psi)\right\|_p \\ &\quad\leqslant\left(\frac{1}{4} \alpha_{dn}(d)^2+\frac{1}{2(dn+1)^3}\right)\|P(D)^2 \psi\|_p \qquad (6.5)\end{aligned}$$

with $\psi=M_n^2 f$. We can write using Theorem 3.1

$$\begin{aligned}\|P(D)^2 M_n^2 f\|_p &\leqslant 2nd\,\|P(D)\, M_n f\|_p \\ &\leqslant 2nd\,\|P(D)(\tfrac{1}{2}(M_{nd} M_n^2 f+M_n^2 f))\|_p \\ &\quad+nd[\,\|P(D)(M_{dn} M_n^2 f-M_n f)\|_p \\ &\quad+\|P(D)(M_n-I)\, M_n f\|_p] \\ &\leqslant 2nd\,\|P(D)\, g\|_p+(2nd)^2\,\|M_n f-f\|_p \\ &\quad+2n^2 d^2\|\, M_{dn} f-f\|_p .\end{aligned}$$

(Recall $P(D)(M_{dn} M_n^2 f-M_n f)=P(D)\, M_n(M_n f-f)+P(D)\, M_n^2(M_{dn} f-f)$.) We now complete the proof using (6.5) with $\psi=M_n^2 f$ and the above to write

$$\alpha_{dn}(d)\,\|P(D)\,g\|_p \leqslant \|M_{dn}M_n^2 f - M_n^2 f\|_p + \left(\frac{1}{4}\alpha_{nd}(d)^2 + \frac{1}{2(dn+1)^3}\right)$$
$$\times \|P(D)^2 M_n^2 f\|_p$$
$$\leqslant 2\,\|M_{dn}f - f\|_p + 2\,\|M_n f - f\|_p + \left(\frac{1}{2}\alpha_{dn}(d) + \frac{1}{(dn+1)^2}\right)$$
$$\times \|P(D)\,g\|_p.$$

Since $1/d(n+1) \leqslant \alpha_{dn}(d) \leqslant 1/(dn+1)$, we have

$$\frac{1}{n}\|P(D)\,g\|_g \leqslant 8d(2\,\|M_{dn}f - f\|_p + 2\,\|M_n f - f\|_p), \qquad \text{for} \quad n \geqslant 3.$$

To prove (6.4) we choose $g = \frac{1}{2}(M_n^{r+2}f + M_n^{r+1}f)$ with $r = r(p, d)$ such that (5.5) is satified with $A = 2$ (which is possible for $1 < p < \infty$ and any d by Corollary 5.3). Obviously,

$$\|f - g\|_p \leqslant \tfrac{1}{2}(\|M_n^{r+2}f - f\|_p + \|M_n^{r+1}f - f\|_p) \leqslant \tfrac{1}{2}(2r+3)\,\|M_n f - f\|_p.$$

To estimate $(1/n)\,\|P(D)\,g\|$, we use Theorem 4.1 and write

$$\left\|M_n(M_n^{r+1}f) - M_n^{r+1}f - \frac{\alpha_n(d)}{2}P(D)(M_n^{r+2}f + M_n^{r+1}f)\right\|_p$$
$$\leqslant \left(\frac{1}{4}\alpha_n(d)^2 + \frac{1}{2(n+1)^3}\right)\|P(D)^2 M_n^{r+1}f\|_p$$

and

$$\|P(D)^2 M_n^{r+1}f\|_p \leqslant 2n\,\|P(D)\,M_n f\|_p$$
$$\leqslant 2n\,\|P(D)(\tfrac{1}{2}M_n^{r+2}f + \tfrac{1}{2}M_n^{r+1}f)\|_p$$
$$+ n \cdot 2\,dn(\|M_n^{r+1}f - f\|_p + \|M_n^r f - f\|_p)$$
$$\leqslant 2n\,\|P(D)\,g\|_p + n^2 2\,d(2r+1)\,\|M_n f - f\|_p$$

and proceed as before to complete the proof. ∎

THEOREM 6.3. *Under the assumptions of Theorem* 6.1, *we have*

$$\|M_n f\|_p \sim K(f, 1/n)_p$$

for $1 \leqslant p \leqslant \infty$ *and* $d = 1, 2, 3$.

Proof. Actually, we only have to prove the equivalence for $p=1$ and $p=\infty$ in case $d=2$ and $d=3$. We choose $g=\frac{1}{2}(M_n^4 f+M_n^3 f)$ and use (4.1) to write

$$\left\|M_n\psi-\psi-\frac{\alpha_n(d)}{2}P(D)(M_n\psi+\psi)\right\|_p\leqslant\left(\frac{1}{4}\alpha_n(d)^2+\frac{1}{2(n+1)^3}\right)\|P(D)^2\psi\|_p$$

with $\psi=M_n^3 f$. The proof now follows the same lines (see also [8]) using the fact that $n\alpha_n(d)$ is close to one for $n\geqslant n_0$ and using Theorem 3.2 instead of Theorem 3.1. ∎

Remark 6.4. It would be desirable to prove Theorem 6.3 for all d and we believe that this result is valid. This would follow from the estimate

$$\|P(D)\,M_n^r f\|_p\leqslant\varepsilon(r)\,n\,\|f\|_p$$

with $\varepsilon(r)=o(1)$, $r\to\infty$. While we believe this last estimate to be true, we are not able to prove it at present for $p=1$ and $p=\infty$.

7. Iterations

In this section we use the results of the last section to obtain theorems about equivalence to $K_r(f,t^r)$.

THEOREM 7.1. *For $f\in L_p(T)$, $1<p<\infty$, or $f\in L_p(T)$, $\dim T\leqslant 3$ and $1\leqslant p\leqslant\infty$,*

$$K_r(f,n^{-r})_p\sim\|(M_n-I)^r f\|_p, \tag{7.1}$$

where $K_r(f,t^r)_p$ is given by (6.1) *and M_n by* (2.1).

Proof. The estimate

$$K_r(f,n^{-r})_p\leqslant C(r)\,\|(M_n-I)^r f\|_p \tag{7.2}$$

follows from the estimate achieved in Theorems 6.1 and 6.3 and Theorem 10.4 of [8] using the estimate

$$\frac{1}{n}\|P(D)(M_n^l f)\|_p\leqslant B\,\|f-M_n f\|_p \tag{7.3}$$

for some l. We proved for some r

$$\frac{1}{2n}\|P(D)(M_n^{r+1}f+M_n^r f)\|_p\leqslant B_1\,\|f-M_n f\|_p,$$

which implies (7.3) (Equation (7.3) could have been proved directly.) The estimate

$$\|(M_n - I)^r f\|_p \leqslant B(r)\, K_r(f, n^{-r})_p \tag{7.4}$$

was shown when proving Theorem 4.1 of [2] and is the easier direction in any case. ∎

We can also prove the following result which is of interest only for $p=1$ and $p=\infty$ when $d>3$, as otherwise it is just a special case of Theorem 7.1.

THEOREM 7.2. *For $f \in L_p(T)$, $1 \leqslant p \leqslant \infty$, we have*

$$K_r(f, n^{-r})_p \sim \max_{0 \leqslant i \leqslant r} \|(M_n - I)^{r-i} (M_{nd} - I)^i f\|_p, \qquad n \geqslant n_0, \tag{7.5}$$

and

$$K_r(f, n^{-r})_p \sim \|(M_n - I)^r f\|_p + \|(M_{nm} - I)^r f\|_p, \qquad n \geqslant n_0 \tag{7.6}$$

for some $m = m(r)$.

Remark 7.3. The advantage of (7.5) is that it is easier to prove (and d may be smaller than m). The advantage of (7.6) is that it yields two terms and hence the iteration is still a strong converse inequality of type B in the terminology of [8]. Moreover, M_{nd} and M_{nm} in (7.5) and (7.6) can be replaced by M_l, with $nd \leqslant l \leqslant nA$ and $nm \leqslant l \leqslant nA$, respectively.

Proof of Theorem 7.2. The direct inequalities in (7.5) and (7.6), that is,

$$\|(M_n - I)^{r-i} (M_{nd} - I)^i f\|_p \leqslant C K_r(f, n^{-r})_p, \qquad 0 \leqslant i \leqslant r,$$

and

$$\|(M_{sn} - I)^r f\|_p \leqslant C K_r(f, n^{-r})_p, \qquad s = 1, m,$$

follows from earlier results (see for instance the proof of Theorem 4.1 in [2]). For the proof of (7.5) we have to show

$$K_r(f, n^{-r})_p \leqslant B \max_{0 \leqslant i \leqslant r} \|(M_n - I)^{r-i} (M_{nd} - I)^i f\|_p. \tag{7.7}$$

To obtain (7.7) we choose g as

$$\begin{aligned} g \equiv O_{n,r} f &= \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} O_n^s f, \\ O_n f &\equiv \tfrac{1}{2}(M_{nd} M_n^2 + M_n^2) f. \end{aligned} \tag{7.8}$$

We estimate $\|f - g\|_p$ by

$$\begin{aligned} \|f - g\|_p = \|f - O_{n,r} f\|_p &= \|(O_n^r - I)^r f\|_p \leqslant r^r \|(O_n - I)^r f\|_p \\ &\leqslant A r^r \max_{0 \leqslant i \leqslant r} \|(M_n - I)^{r-i} (M_{nd} - I)^i\|_p. \end{aligned}$$

To complete the proof of (7.7) we estimate $n^{-r} \|P(D)^r g\|_p$ by

$$\begin{aligned} n^{-r} \|P(D)^r g\|_p &= n^{-r} \|P(D)^r O_{n,r} f\|_p \leqslant 2^r n^{-r} \max_{1 \leqslant s \leqslant r} \|P(D)^r O_n^{rs} f\|_p \\ &\leqslant 2^r n^{-r} \|P(D)^r O_n^r f\|_p \\ &\leqslant A n^{-r+1} (\|P(D)^{r-1} O_n^{r-1}(M_n - I) f\|_p \\ &\quad + \|P(D)^{r-1} O_n^{r-1}(M_{nd} - I) f\|_p) \\ &\leqslant \cdots \leqslant B \max_{0 \leqslant i \leqslant r} \|(M_n - I)^{r-i} (M_{nd} - I)^i f\|_p. \end{aligned}$$

To prove (7.6) it remains to show that for some integer m we have

$$K_r(f, n^{-r})_p \leqslant B(\|(M_n - I)^r f\|_p + \|(M_{nm} - I)^r f\|_p). \tag{7.9}$$

We postpone the choice of m and choose g as

$$g = \sum_{s=1}^{r} (-1)^{s+1} \binom{r}{s} M_n^{(r+1)s} f. \tag{7.10}$$

The estimate of $\|f - g\|_p$ is given by

$$\|f - g\|_p = \|(M_n^{r+1} - I)^r f\|_p \leqslant (r+1)^r \|(M_n - I)^r f\|_p.$$

To estimate $n^{-r} \|P(D)^r g\|_p$ we write

$$\begin{aligned} n^{-r} \|P(D)^r g\|_p &\leqslant n^{-r} 2^r \sup_{1 \leqslant s \leqslant r} \|P(D)^r M_n^{(r+1)s} f\|_p \\ &\leqslant n^{-r} 2^r \|P(D)^r M_n^{r+1} f\|_p, \end{aligned}$$

and hence it is sufficient to estimate $n^{-r} \|P(D)^r M_n^{r+1} f\|_p$. Using Theorem 4.3 with mn replacing n, and m chosen so that $2r\, d 2^r \leqslant m$, we have

$$\begin{aligned} &\|(M_{nm} - I)^r M_n^{r+1} f - \alpha_{nm}(d)^r P(D)^r M_n^{r+1} f\|_p \\ &\leqslant \frac{r}{2(nm)^{r+1}} \|P(D)^{r+1} M_n^{r+1} f\|_p \\ &\leqslant \frac{r\,d}{m} \frac{1}{(mn)^r} \|P(D)^r M_n^r f\|_p \\ &\leqslant \frac{r\,d}{m} \frac{1}{(mn)^r} \|P(D)^r M_n^r (M_n - I)^r f\|_p \\ &\quad + \frac{r\,d}{m} \frac{1}{(mn)^r} \sum_{s=1}^{r} \binom{r}{s} \|P(D)^r M_n^{r+s} f\|_p \\ &\leqslant \frac{r\,d}{m} \frac{2^r}{(mn)^r} \|P(D)^r M_n^{r+1} f\|_p + \frac{r}{2} \left(\frac{2d}{m}\right)^{r+1} \|(M_n - I)^r f\|_p. \end{aligned}$$

Since $r\,d\,2^r/m \leqslant 1/2$, we complete the proof writing

$$\alpha_{nm}(d)^r \, \|P(D)^r M_n^{r+1} f\|_p \leqslant \frac{1}{2} \frac{1}{(mn)^r} \|P(D)^r M_n^{r+1} f\|_p + \|(M_{nm} - I)^r f\|_p + \frac{r}{2} \left(\frac{2d}{m}\right)^{r+1} \|(M_n - I)^r f\|_p$$

and recalling $\alpha_{nm}(d)^r = (1/nm)^r + O(n^{-r-1})$. ∎

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